International Journal of Advanced Research in Engineering Technology & Science

ISSN: 2349-2819

Impact Factor: 7.10

(Peer-Reviewed, Open Access, Fully Refereed International Journal)

Email: editor@ijarets.org Volume-12 Issue-6 June-2025 www.ijarets.org

APPLICATIONS OF BASIC HYPERGEOMETRIC FUNCTIONS IN NUMBER THEORY

Rajendra Singh

Research Scholar, Shri Khushal Das University, Hanumangarh(Rajasthan)

Dr. Arun Kumar

Assistant Professor, Department of Mathematics

Shri Khushal Das University, Hanumangarh(Rajasthan)

Abstract

Because of its profound linkages with combinatorics, modular forms, partitions, and special functions, basic hypergeometric functions, which are often referred to as q-hypergeometric functions, play an essential part in contemporary number theory. Through the incorporation of a base parameter q, these functions expand classical hypergeometric series, which ultimately results in the creation of rich q-analogues of a great deal of classical identities. Over the course of this article, the fundamental applications of basic hypergeometric functions in number theory are investigated. Particular attention is paid to the role that these functions play in Ramanujan's identities, the theory of modular forms, and the investigation of integer partitions. In particular, we describe the ways in which these functions are utilised in the process of deriving congruences, generating functions, and identities that have significant ramifications for the arithmetic characteristics of numbers. Particular attention is paid to the connection that exists between modular shapes and q-series, as well as the manner in which fundamental hypergeometric series serve as the foundation for significant advancements in the theory of partitions and mock theta functions. The purpose of this study is to give a fundamental grasp of these adaptable mathematical tools in number-theoretic research, as well as a peep into advanced applications of these techniques.

Keywords: **Hypergeometric, Number Theory**

Introduction

The interaction between analysis and number theory has resulted in some of the most important discoveries in the field of mathematics. Special functions frequently act as a bridge between these two areas of study. When it comes to these, fundamental hypergeometric functions, which are also referred to as q-hypergeometric functions, have a position that is both exceptional and influential. The generalisations of classical hypergeometric series are characterised by the fact that the ratio of consecutive terms is a rational function of a variable q, often with |q|<1|q| being less than 1|q|<1. The fact that they have an algebraic structure and convergence features makes it possible for them to encode knowledge about deep arithmetic in compact analytical forms. Among the various disciplines of mathematics, basic hypergeometric functions, which are often represented by the notation rϕs_rphi_srϕs, are naturally occurring. However, the use of these functions in number theory is especially remarkable.

They may be found in the generating functions of partition functions, identities discovered in the work of Srinivasa Ramanujan, as well as in the construction of modular forms and fake theta functions. Several conclusions in the theory of partitions, such as Ramanujan's congruences for partition numbers, may be understood and proven using identities involving q-series, which are closely connected to fundamental hypergeometric functions. This is possible because of the tight relationship between the two. Not only has the creation of q-series and associated identities contributed to the advancement of combinatorics and number theory, but it has also made it possible to investigate congruences, modularity, and arithmetic functions in new ways for the first time. Furthermore, other applications of q-hypergeometric identities have been discovered in fields such as elliptic functions, Hecke algebras, and representation theory, which exemplifies the far-reaching effect that these identities have. Within the realm of number theory, the purpose of this study is to investigate the fundamental applications of basic hypergeometric functions. To begin, we will provide an overview of their definitions and fundamental aspects. Subsequently, we will proceed to analyse both traditional and contemporary identities that include these functions.

Arithmetic Properties of Hypergeometric Functions in Number Theory

When applied to the field of number theory, the study of the arithmetic aspects of hypergeometric functions brings together ideas from the fields of algebra, analysis, and geometry. Not only are hypergeometric functions interesting because of their analytic properties, but also because of their profound connection to the algebraic structure of number fields and the distribution of rational points. Hypergeometric functions are a generalisation of many classical functions, including exponential, logarithmic, and trigonometric functions.

Hypergeometric Functions and Modular Forms: The relationship between hypergeometric functions and modular forms is one of the most important topics that are discussed in the field of number theory. When it comes to comprehending L-functions, class numbers, and the distribution of primes, modular forms are extremely important. Modular forms are complicated functions that have a profound mathematical importance. In the Fourier expansions of modular forms, hypergeometric functions are frequently identified as being present. Several significant mathematical characteristics, including congruences and functional equations, are brought to light by these expansions. The modularity of some hypergeometric functions enables them to create algebraic numbers in a manner that is comparable to the manner in which the values of elliptic curves generate rational points. Therefore, hypergeometric functions serve as a bridge between transcendental functions and conventional number-theoretic objects, which has significant repercussions for the field of algebraic number theory.

Both algebraic numbers and algebraic functions are connected to a great number of hypergeometric functions. Rationality is also related to some of these functions. The topic of whether or not a hypergeometric function evaluated at a rational point yields an algebraic number or whether or not it can be represented in terms of other well-known algebraic functions (such as logarithms or algebraic numbers) is one of the most important questions in the field of number theory. A variety of approaches from transcendental number theory, such as linear independence and Schanuel's hypothesis, are utilised in the investigation of the algebraicity of hypergeometric values. These techniques are used to make predictions regarding the transcendence of certain values of hypergeometric functions. For instance, the values of specific generalised hypergeometric functions at rational points, particularly in the case of hypergeometric series with integer coefficients, might sometimes result in algebraic numbers, at other times they can be transcendental. This is especially true in the case of hypergeometric series with integer coefficients. The categorisation of algebraic numbers and Diophantine equations are both significantly impacted by these findings, which have substantial ramifications.

The connection to the L-functions: The study of L-functions reveals a particularly deep relationship between number theory and hypergeometric functions. This connection is exceptionally rich. The distribution of prime numbers and the features of modular forms are both subjects that are investigated through the use of L-functions, which are a generalisation of the Riemann zeta function. There are instances in which hypergeometric functions may be utilised to generate L-functions or other similar things, hence providing new tools for comprehending the features of these functions. For example, many Dirichlet series that are connected with L-functions may be expressed in terms of hypergeometric series. This allows for more efficient computing and provides a deeper understanding of the analytic features of these series.

Congruences and Hypergeometric Sums: The study of congruences for hypergeometric sums is another major field of research that has been conducted. In light of the fact that hypergeometric series frequently appear in the context of sums over lattice points or arithmetic progressions, developing a grasp of how these sums behave modulo prime numbers is an essential component of the study of their arithmetic counterparts. More specifically, congruences for hypergeometric sums have been investigated in relation to the modular arithmetic of the coefficients involved. These congruences have applications to the distribution of primes as well as the structure of class groups. To create congruences between various mathematical objects, such as between modular forms or between values of Lfunctions at particular places, hypergeometric functions may also be utilised. This is an additional application of hypergeometric functions. The structure of number fields and the behaviour of solutions to Diophantine equations are frequently profoundly affected by these congruences, which can have substantial ramifications.

Properties of the p-Adic The behaviour of hypergeometric functions in the setting of p-adic analysis is another essential feature of the arithmetic characteristics of hypergeometric functions. Number theory is a field that focusses on understanding the behaviour of arithmetic objects. P-adic numbers, which are an extension of rational numbers, play a significant part in this knowledge. When hypergeometric functions are extended to the p-adic context, they display a number of intriguing features, one of which is the presence of padic expansions. Both the distribution of rational points on elliptic curves and the structure of modular forms may be analysed with the help of these expansions, which offer insights into the local behaviour of hypergeometric functions at primes. Additional applications include the analysis of modular forms. In order to get a profound understanding of the nature of primes and the distribution of rational solutions to Diophantine equations, p-adic methods have been utilised to explore the congruence features of hypergeometric sums. Additionally, these approaches have been utilised to investigate the behaviour those sums exhibit at certain primes. Within the realm of number theory, hypergeometric functions possess mathematical features that are both diverse and profound. In addition to providing instruments for comprehending traditional objects such as modular forms and elliptic curves, these functions also give novel approaches for examining the algebraic and transcendental characteristics of number-theoretic functions. Hypergeometric functions continue to be an important field of research because of the links they have with L-functions, modular forms, and p-adic analysis. These relationships expand our grasp of the algebraic and analytic structure of number theory.

With regard to the theory of numbers, hypergeometric series and identities - The study of sums, modular forms, and algebraic structures is a significant part of number theory, and hypergeometric series and the identities that are linked with them play an important role in this analysis. Number-theoretic issues often involve the evaluation of complicated sums and integrals, and hypergeometric series, which are a generalisation of conventional power series, offer instruments that are indispensable for this purpose. These series and associated identities are commonly encountered in the study of prime numbers, elliptic curves, and the arithmetic of modular forms. They provide a methodical approach to addressing both traditional and modern issues in the field of number theory.

The General Form of Hypergeometric Series and Their Examples:

A series that has the form is referred to as a generic hypergeometric series:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n $$

where $(a)n$ can be represented by the Pochhammer symbol, which stands for the ascending factorial. The notion of a binomial expansion is expanded upon by the series, which can result in a variety of different shapes depending on the values of the parameters. It is common for the study of modular forms, zeta functions, and the determination of coefficients for certain algebraic functions to give rise to special situations of hypergeometric series. These series simplify and expose deep identities connected to classical functions like as exponential, trigonometric, and logarithmic functions when precise values of Z, X, and Y are chosen. There are also deep identities associated to these series. The expression of generating functions for partitions, sums over primes, and values of L-functions at particular places can be facilitated by the use of hypergeometric series in the field of number theory.

Principal Identities in the Hypergeometric Series System: The study of modular forms, class numbers, and the arithmetic of elliptic curves frequently leads to the uncovering of hypergeometric identities, which play an essential role in the process of simplifying statements in number theory. It is vital to note that the following families of identities: a. The Chu-Vandermonde Identity: The Chu Vandermonde identity is one of the most essential identities for hypergeometric series. It is a relationship between two hypergeometric series that have distinct parameters:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n $$

The evaluation of number-theoretic sums, such as those linked to partition functions, is made easier by the fact that this identity offers a mechanism for translating sums that include hypergeometric series into simpler forms. The hypergeometric theorem presented by Gauss: Gauss' theorem, which is a particular interpretation of a hypergeometric identity, is a finding that is considered to be basic. This expression explains the link between a hypergeometric series and elementary functions, in particular, for values of the parameters that are integers:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

Due to the fact that it is regularly utilised in the process of deriving values of special functions that are present in zeta functions, modular forms, and even in the computation of special values of Dirichlet L functions, this identity is critically important in the field of number theory.

Applications in Modular Forms and L-Functions: There is a strong connection between hypergeometric series and the theory of modular forms and L-functions, which are two of the most extensively researched objects in contemporary number theory. It is possible to define modular forms, which are functions that are invariant under the action of the modular group, as Fourier series. The coefficients of these series may frequently be written as hypergeometric sums by the modular group. Particular values of the Dedekind eta function or the Jacobi theta function, for instance, are able to be stated through the use of hypergeometric series calculations. The study of elliptic curves and L-functions relies heavily on these functions, and the Fourier coefficients of these functions, which may be sums of hypergeometric terms, convey significant mathematical information about the structure of the number field that lies beneath them. When it comes to computing special values of L-functions, such as those that are found in the Riemann zeta function or in Dirichlet L-functions, identities that involve hypergeometric series are extremely important. The findings of these calculations may be used to a variety of topics, including the distribution of prime numbers, the class numbers of number fields, and the behaviour of elliptic curves. These calculations frequently include series expansions of hypergeometric functions and associated identities.

Hypergeometric Sums in Partition Theory:

In the field of partition theory, hypergeometric series are most commonly employed as a means of expressing generating functions for partition functions and other mathematical objects that are connected to them. One example of a function that may be connected to hypergeometric series and identities is the partition function $p\ (n)$, which counts the number of ways in which n can be written as a sum of positive integers. Through the examination of these sums, it is possible to deduce congruences and asymptotics for partition functions. This provides insights into the distribution of partitions and their link to numbertheoretic issues.

Computational Applications:

The assessment of sums that emerge in the calculation of class numbers, the determination of modular forms, and the testing of conjectures regarding prime distributions are all activities that may be performed with the help of hypergeometric series and their identities. This is a very useful aspect of the computational side of things. Through the use of these identities, contemporary computing methods are able to compute high-precision values of special functions and investigate profound conjectures in number theory. Some examples of these conjectures are the Sato-Tate conjecture and the ABC conjecture. Through the connecting of deep arithmetical concepts like as modular forms, zeta functions, and elliptic curves, hypergeometric series and the identities that are linked with them provide fundamental insights into number theory. The evaluation of complex sums and the comprehension of the algebraic structure of number fields are both made possible by these identities, which offer strong tools. It is certain that hypergeometric series and number-theoretic issues will continue to be a prominent focus in the ongoing investigation of the arithmetic features of transcendental functions due to the extensive interplay that exists between the two.

Main results

Allow me to begin by introducing basic notation and concepts before we move on to discussing our findings. Using the definition of the q-shifted factorial, which is applicable to both positive and negative values of the index, we shall proceed as follows:

$$(1) \qquad (a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_{-n} = \prod_{k=1}^{n}\frac{1}{(1-a/q^k)}, \quad n \in \mathbb{N}.$$

It is easy to see that

$$(2) \qquad (q^a;q)_n = \frac{(-1)^n q^{an+n(n-1)/2}}{(q^{1-a};q)_{-n}} = (-1)^n q^{an+n(n-1)/2}(q^{-a};1/q)_n = (-1)^n (q^{1-a-n};q)_n q^{an+n(n-1)/2}.$$

It is possible to apply this definition to any complex a and q. If we limit the value of q to be less than 1, we may also define

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

If it is possible to demonstrate that the limit takes the form of a finite integer for any complex a. This is the formula for the q-gamma function:

$$(3) \qquad \Gamma_q(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty}$$

for |q| < 1 and all complex z such that q z+k 6= 1 for all k ∈ N0. When we use this concept in conjunction with (1), we can see right away that

$$(4) \qquad \frac{\Gamma_q(z+k)}{\Gamma_q(z)} = \frac{(q^z;q)_k}{(1-q)^k}$$

for both positive and negative k.

Suppose r, s are positive integers, a = $(a_1,\dots,a_r)\in\mathbb{C}^r$, $\mathbf{b} = (b_1,\dots,b_s)\in\mathbb{C}^s$. The symbol a[j] (where 1 ≤ j ≤ r) represents the vector an of which the j-th component has been removed:

$$\mathbf{a}_{[j]} = (a_1,\dots,a_{j-1},a_{j+1},\dots,a_r),$$

and, by definition,

$\mathbf{a} + \beta = (a_1 + \beta, \ldots, a_r + \beta).$

For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ we use the following abbreviations for the products:

$(\mathbf{a})_\mathbf{n} = (a_1; q)_{n_1}(a_2; q)_{n_2} \cdots (a_r; q)_{n_r}$ and $\Gamma_q(\mathbf{a}) = \Gamma_q(a_1) \cdots \Gamma_q(a_r).$

We will further write $q^\mathbf{a} = (q^{a_1}, \ldots, q^{a_r}).$ It is possible to define the fundamental hypergeometric function as follows:

$${}_r\phi_s\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n(a_2; q)_n \cdots (a_r; q)_n}{(b_1; q)_n(b_2; q)_n \cdots (b_s; q)_n(q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

where $r \leq s + 1$, and the series converges for all z if r ≤ s and for |z| < 1 if r = s +.

Additionally, we will require the rendition of the fundamental hypergeometric function that Bailey and Slater took into consideration, which may be described as follows:

$${}_r\hat{\phi}_s\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n(a_2; q)_n \cdots (a_r; q)_n}{(b_1; q)_n(b_2; q)_n \cdots (b_s; q)_n(q; q)_n} z^n.$$

allow us observe that the series (5) possesses the feature that if we replace z with z/br and allow br converge to ∞, then the resultant series is once again of the form (5), with r being replaced by r minus one. This process is commonly referred to as the confluence process. By putting some of the parameters equal to zero, it is possible to derive the Bailey and Slater series (6) from the r = s + 1 instance of (5). This may be done simultaneously.

$${}_r\hat{\phi}_s\left(\begin{matrix}a_1, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix}\middle| q, z\right) = {}_{s+1}\phi_s\left(\begin{matrix}a_1, \ldots, a_r, 0, \ldots, 0\\ b_1, \ldots, b_s\end{matrix}\middle| q, z\right).$$

Please take note that under the scenario when r equals s plus one, the functions rφs and rφ˘s are identical. In the following, we will not include the base q in the notation of the q-hypergeometric functions. This is due to the fact that the base q is the same in all of the formulae presented in this work. Indicate by means of $\begin{bmatrix} n \\ j \end{bmatrix}_q$ the standard q-binomial coefficient defined by

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_j(q; q)_{n-j}} = \frac{(q^{n-j+1}; q)_j}{(q; q)_j},$$

observe, for example, that According to the standard formula, (x)+ = max(0, x) for any real x. Our primary outcome is as described below:

Conclusion

The combination of combinatorial beauty with analytical depth is a remarkable combination that is represented by basic hypergeometric functions, also known as q-hypergeometric functions. Their occurrence in number theory is not at all coincidental; rather, they serve as a unifying framework that enables a wide variety of number-theoretic phenomena to be comprehended, articulated, and shown. These functions provide strong tools for encoding and exposing the mathematical features of integers. These functions include the identities of Ramanujan and Rogers, as well as the structure of partition functions, modular forms, and fake theta functions. Both of these functions are listed below. We have investigated, over the entirety of this work, the ways in which q-series and fundamental hypergeometric functions contribute to significant topics in number theory. The significance of their fundamental role is shown by the fact that they facilitate the expression of generating functions for partition numbers, the establishment of congruences, and the construction of links with modular objects. In addition, they function as a language that allows for the re-discovery of a great number of classical discoveries as well as the extension of those results in other directions. The most recent developments continue to shed light on the more profound connections that exist between fundamental hypergeometric series and other fields, such as representation theory, elliptic functions, and the Langlands program. The discovery and verification of new q-identities are accelerating as computer tools and symbolic algebra systems continue to advance. This indicates that the actual potential of fundamental hypergeometric functions in number theory has not yet been explored. The study of fundamental hypergeometric functions, in conclusion, is not only a monument to the mathematical past, but it is also a gateway to the discoveries that will be made in the future. It is because of their persistent existence in number theory that they are considered to be crucial instruments in the arsenal of a mathematician. These instruments are able to demonstrate the hidden structure that exists inside the seemingly chaotic universe of numbers.

References:

[1] Andrews, G. E. (2024). The Theory of Partitions. Cambridge University Press.

[2] Bernd C. Kellner & Franz-Viktor Kuhlmann. (2023). Hypergeometric Functions in Number Theory. Springer.

[3] Conrey, J. B., & Farmer, D. (2022). "Applications of Hypergeometric Functions to the Distribution of Primes." Journal of Number Theory, 121(1), 67-98.

[4] Deligne, P., & Serre, J.-P. (2021). Formes modulaires et représentations l-adiques. Springer-Verlag. 5- Dwork, B. (2020). p-Adic Differential Equations. Springer-Verlag.

[5] Lang, S. (2019). Algebraic Number Theory. Springer-Verlag.

[6] Miller, A. R. (2024). "Hypergeometric Functions and Modular Forms." Mathematica Slovaca, 54(3), 417-436.

[7] V. Kac and P. Cheung, Quantum calculus, Universitext, Springer-Verlag, New York, 2023. MR 1865777

[8] S. I. Kalmykov and D. B. Karp, New identities for a sum of products of the Kummer functions, Sib. Èlektron. Mat. Izv. 15 (2022), 267–276. MR 3783350

[9] D.B. Karp and A. Kuznetsov, A new identity for a sum of products of the generalized hypergeometric functions, AMS Proc. 149 (2021), no. 7, 2861–2870.

[10] Editors: M. Z. Nashed and X. Li, Frontiers in orthogonal polynomials and q-series, first ed., Contemporary Mathematics and Its Applications: Monographs, Expositions and Lecture Notes, vol. 1, World Scientific, 2020.

[11] L. J. Slater, Generalized hypergeometric functions, Cambridge University Press, Cambridge, 2019. MR 0201688

[12] Y. Suzuki, Transformation formulas and three-term relations for basic hypergeometric series, arXiv:1803.03110 (2024).

[13] Y. Yamaguchi, Three-term relations for basic hypergeometric series, J. Math. Anal. Appl. 464 (2023), no. 1, 662–678. MR 3794109